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COMPLEX CONVEXITY AND MONOTONICITY IN QUASI-BANACH LATTICES

BY

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ABSTRACT

In this paper we study the monotonicity and convexity properties in quasi-Banach lattices. We establish relationship between uniform monotonicity, uniform $\mathbb{C}$-convexity, H- and PL-convexity. We show that if the quasi-Banach lattice E has α-convexity constant one for some $0 < \alpha < \infty$, then the following are equivalent: (i) E is uniformly PL-convex; (ii) E is uniformly monotone; and (iii) E is uniformly $\mathbb{C}$-convex. In particular, it is shown that if E has α-convexity constant one for some $0 < \alpha < \infty$ and if E is uniformly $\mathbb{C}$-convex of power type then it is uniformly H-convex of power type. The relations between concavity, convexity and monotonicity are also shown so that the Maurey–Pisier type theorem in a quasi-Banach lattice is proved.

Finally we study the lifting property of uniform PL-convexity: if E is a quasi-Köthe function space with α-convexity constant one and X is a continuously quasi-normed space, then it is shown that the quasi-normed Köthe-Bochner function space $E(X)$ is uniformly PL-convex if and only if both E and X are uniformly PL-convex.

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1. Introduction and Preliminaries

The notion of uniform monotonicity of lattices was first studied by Birkhoff in [3] and its characterization and relations with uniform convexity in Banach function spaces have been further studied in several papers (cf. [13, 14, 15, 16]).

The strict $\mathbb{C}$-convexity of complex Banach space was introduced by Thorp and Whitely in [31] by the corresponding property characterizing the strong maximum modulus theorem for the Banach space-valued analytic functions. For the basic properties and characterizations of $\mathbb{C}$-convexity in certain Banach spaces, see [4, 5, 18, 19, 20].

The uniform version of complex convexity (it is called uniformly $\mathbb{C}$-convex) was studied by Globevnik in [12], and there it was shown that L^1-space is uniformly $\mathbb{C}$-convex, which shows that complex convexity is quite different from the real convexity. For the characterizations of uniform $\mathbb{C}$-convexity in various function spaces, consult [4, 5].

The moduli of complex convexity of complex quasi-Banach spaces and notion of uniform PL-convexity were introduced by Davis, Garling and Tomczak-Jagermann in [7]. In the same paper, it was shown that $L^p(X)$ $(0 < p < \infty)$ is uniformly PL-convex if the continuously quasi-normed space X is uniformly PL-convex. It was shown by Dilworth in [10] that a complex Banach space X is uniformly PL-convex if and only if it is uniformly $\mathbb{C}$-convex. Notice that this equivalence does not hold in certain quasi-Banach lattices. In fact uniform $\mathbb{C}$-convexity does not imply uniform PL-convexity [24, 28].

Another notion of complex convexity (it is called uniform H-convexity) was introduced by Xu in [32]. He showed in [34] that a complex quasi-Banach lattice is uniformly PL-convexifiable if and only it is uniformly H-convexifiable.

Recently, it has been shown by Hudzik and Narloch [17] that a Köthe function space is uniformly monotone if and only if it is uniformly $\mathbb{C}$-convex. This result was extended to the case of complex Banach lattices by the author in [23]. In the same paper, the lifting properties of uniform $\mathbb{C}$-convexity was also investigated. It was proved that a Köthe function space E is uniformly $\mathbb{C}$-convex and a Banach space X is uniformly $\mathbb{C}$-convex if and only if the Köthe-Bochner function space $E(X)$ is uniformly $\mathbb{C}$-convex. For the lifting property of complex geometric properties to Lebesgue–Bochner spaces $L^p(X)$ $(0 < p < \infty)$, we refer to [7, 11].

In this paper, we shall study the properties of moduli of uniform monotonicity of complex quasi-Banach lattices and their relations with various complex convexities. First we introduce some notation and terminology.

Let $\mathbb{F}$ be a (real or complex) scalar filed. Recall that a **quasi-norm** on a vector space X over $\mathbb{F}$ is a real non-negative function $\|\cdot\|$ on X satisfying

(1) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all scalars $\alpha \in \mathbb{F}$ and all x in X;

(2) there exists $K > 0$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all x and y in X; and

(3) $\|x\| = 0$ if and only if $x = 0$.

The smallest K for which (2) holds is called the **quasi-norm constant** of $(X, \|\cdot\|)$. The complete quasi-normed space X is called **quasi-Banach space**. X is said to be α**-normable**, where $0 < \alpha \leq 1$, if for some constant B we have

$$\|x_1 + \cdots + x_n\| \leq B(\|x_1\|^\alpha + \cdots + \|x_n\|^\alpha)^{1/\alpha}, \tag{1.1}$$

for any $x_1, \ldots, x_n$ in X. If inequality (1.1) holds for $B = 1$, the quasi-norm $\|\cdot\|$ is called an α**-norm**.

In a real vector lattice E, we use the standard notation: let A be a subset of E and let x, y be two elements in E,

(1) $x \vee y := \sup\{x, y\}$,
$\bigvee_{x\in A} x := \sup A$, if $\sup A$ exists in E;

(2) $|x| := x \vee (-x)$;

(3) $x^+ := x \vee 0$, $x^- := (-x) \vee 0$;

(4) $x \wedge y := \inf\{x, y\}$ and $\bigwedge_{x\in A} x := \inf A$, if $\inf A$ exists in E.

Now let $(E, \|\cdot\|)$ be a **quasi-Banach lattice**, that is, E is a real vector lattice with a complete quasi-norm $\|\cdot\|$ satisfying the monotonicity condition: for x, y in E

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\|.$$

The Aoki–Rolewicz theorem (see [1, 30]) asserts that a quasi-Banach space X with quasi-norm constant K is α-normable with $B = 4$ in inequality (1.1), where α is defined by the equation $(2K)^\alpha = 2$. We can then have an equivalent quasi-norm

$$|||x||| = \inf\left\{\left(\sum_{i=1}^n \|x_i\|^\alpha\right)^{1/\alpha} : x = x_1 + \cdots + x_n\right\}.$$

Thus $(E, |||\cdot|||)$ is an α-norm. In the case of quasi-Banach lattice E, we can obtain the lattice renorming $|||\cdot|||$ defined as

$$|||x||| = \inf\left\{\left(\sum_{i=1}^n \|x_i\|^\alpha\right)^{1/\alpha} : |x| = x_1 + \cdots + x_n,\ x_1 \geq 0, \ldots, x_n \geq 0\right\},$$

and so that $(E, |||\cdot|||)$ is an α-norm.

We confine ourselves to **continuously quasi-normed spaces** $(X, \|\cdot\|)$, that is, the quasi-norm $\|\cdot\|$ is uniformly continuous on the bounded subsets of X. Notice that every quasi-Banach space with an α-norm is a continuously quasi-normed space.

Let $0 < p < \infty$. A quasi-Banach lattice E is said to be p**-convex** (resp. p**-concave**) if there exists a constant $C > 0$ such that for every finite sequence $x_1, \ldots, x_n$ in E we have

$$\Big\| \Big(\sum_{j=1}^n |x_j|^p \Big)^{1/p} \Big\| \le C \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p}$$

$$\Big(\text{resp. } \Big\| \Big(\sum_{j=1}^n |x_j|^p \Big)^{1/p} \Big\| \ge C^{-1} \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p} \Big).$$

Recall that the Krivine functional calculus, allows us to define the element $(\sum_n |x_n|^p)^{1/p}$ in E analogously, as for Banach lattices (see [6, 26, 21]). The smallest constant C is called the p-convexity (resp. p-concavity) constant of E and is denoted by $M^{(p)}(E)$ (resp. $M_{(p)}(E)$). For $0 < p \le 1$, it is easy to check that the lattice p-convexity implies p-normability and that E is a continuously quasi-normed space if $M^{(p)}(E) = 1$.

Notice that if E is a p-convex (resp. q-concave) quasi-Banach lattice then there is lattice renorming $(E, \|\!|\cdot|\!\|)$ of which the p-convexity (resp. q-concavity) constant is equal to one: for each $x \in E$, define the quasi-norm

$$\|\!|x|\!\| = \inf \Big\{ \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p} : |x| = \Big(\sum_{j=1}^n |x_j|^p \Big)^{1/p}, x_1, \ldots, x_n \in E \Big\}$$

$$\Big(\text{resp. } \|\!|x|\!\| = \sup \Big\{ \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p} : |x| = \Big(\sum_{j=1}^n |x_j|^p \Big)^{1/p}, x_1, \ldots, x_n \in E \Big\} \Big).$$

Now we denote by $E^{(p)}$ the p**-convexification** of a quasi-Banach lattice E (cf. [6, 26]). In the case of function spaces, $E^{(p)}$ can be defined by

$$E^{(p)} = \{x : |x|^p \in E\},$$

with the quasi-norm

$$\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}.$$

If E is α-convex and q-concave, then $E^{(p)}$ is αp-convex and qp-concave with $M^{(\alpha p)}(E^{(p)}) = M^{(\alpha)}(E)^{1/p}$ and $M_{(qp)}(E^{(p)}) = M_{(q)}(E)^{1/p}$. If E is α-convex with $M^{(\alpha)}(E) = 1$, then $E^{(1/\alpha)}$ is 1-convex and $M^{(1)}(E^{(1/\alpha)}) = 1$, so it is a Banach lattice.

The **complexification** $E^{\mathbb{C}}$ of a real quasi-Banach lattice, E, consists of all elements $x + iy$ for $x, y \in E$ with quasi-norm $\|x + iy\| = \|(|x|^2 + |y|^2)^{1/2}\|_E$. Then $E^{\mathbb{C}}$ is a complex quasi-Banach space (see [26]). We call E a **complex quasi-Banach lattice** if it is a complexification of some real quasi-Banach lattice. Throughout the paper, we denote by E a complex quasi-Banach lattice and by X a complex quasi-Banach space if we do not specify otherwise.

The following moduli of complex convexity of complex quasi-Banach space X were introduced in [7]: for $0 < p < \infty$ and $\epsilon \geq 0$, we define

$$H_p^X(\epsilon) = \inf\left\{\left(\int_{\mathbb{T}} \|x + e^{i\theta}y\|^p dm(\theta)\right)^{1/p} - 1 : \|x\| = 1, \|y\| \geq \epsilon\right\}$$

and

$$H_\infty^X(\epsilon) = \inf\{\sup\{\|x + e^{i\theta}y\| : \theta \in \mathbb{T}\} - 1 : \|x\| = 1, \|y\| \geq \epsilon\},$$

where $dm = \frac{1}{2\pi}d\theta$ is the normalized Lebesgue measure on $\mathbb{T} = [0, 2\pi]$.

Let f and g be non-negative, non-decreasing functions on $[0, 1]$. We write $g \preceq f$ if there is $K \geq 1$ such that $g(\epsilon/K) \leq Kf(\epsilon)$ for all $0 < \epsilon < 1/K$, and we write $f \sim g$ if $f \preceq g$ and $g \preceq f$ (f and g are then said to be equivalent at zero). It is well-known that for $0 < p < \infty$, all the moduli H_p^X are equivalent at zero [7] and that there exists an absolute constant $A > 0$ such that for every complex Banach space X and $0 < \epsilon \leq 1$, we have [10],

$$A(H_\infty^X(\epsilon))^2 \leq H_1^X(\epsilon) \leq H_\infty^X(\epsilon). \tag{1.2}$$

A complex quasi-Banach space X is **uniformly $\mathbb{C}$-convex** if $H_\infty^X(\epsilon) > 0$ for all $\epsilon > 0$, and it is said to be **uniformly PL-convex** if $H_p^X(\epsilon) > 0$ for all $\epsilon > 0$ and for some $0 < p < \infty$. By inequalities (1.2), a complex Banach space is uniformly $\mathbb{C}$-convex if and only if it is uniformly PL-convex.

A quasi-Banach space is said to be g-uniformly PL-convex if $H_1^X \succeq g$ holds. If $g(\epsilon) = \epsilon^r$ for some $2 \leq r < \infty$ we say that a quasi-Banach space X is **r-uniformly PL-convex** (or **uniformly PL-convex of power type** ϵ^r). Given $0 < p < \infty$, it can be shown (cf. [7]) that E is r-uniformly PL-convex if and only if there exists $\lambda > 0$ such that

$$\left(\int_{\mathbb{T}} \|x + e^{i\theta}y\|^p \ dm(\theta)\right)^{1/p} \geq (\|x\|^r + \lambda\|y\|^r)^{1/r} \tag{1.3}$$

for all x and y in E. We shall denote the largest possible number of λ by $I_{r,p}(E)$.

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For a complex quasi-Banach X, a function $f\colon \mathbb{D} \to X$ is said to be analytic if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}\ (a_n \in X, n \geq 0),$$

where the series converges uniformly in every compact subset of $\mathbb{D}$. For $0 < p \leq \infty$ we let

$$H^p(X) = \{f\colon \mathbb{D} \to X \text{ analytic} : \|f\|_{H^p(X)} < \infty\},$$

where

$$\|f\|_{H^p(X)} = \sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} \|f(re^{i\theta})\|^p dm(\theta) \right)^{1/p}$$

and

$$\|f\|_{H^\infty(X)} = \sup\{\|f(z)\| : z \in \mathbb{D}\}.$$

It is easy to check that $H^p(X)$ is a quasi-Banach space for $0 < p \leq \infty$. Recall that a quasi-Banach space X is said to have the **analytic Radon–Nikodym property** (*ARNP* for short) if there exists $0 < p \leq \infty$ such that every function $f \in H^p(X)$ has a.e. radial limits on $\mathbb{T}$ in X, namely, if $\lim_{r\to 1} f(re^{i\theta})$ exists a.e. on $\mathbb{T}$ in X.

Another notion used in this paper is the uniform H-convexity [32]. For $0 < p < \infty$ and $0 < \epsilon < \infty$, let

$$h_p^X(\epsilon) = \inf\{\|f\|_{H^p(X)} - 1 : \|f(0)\| = 1, \|f - f(0)\|_{H^p(X)} \geq \epsilon\}.$$

Then X is said to be **uniformly** H_p**-convex** if $h_p^X(\epsilon) > 0$ for every $\epsilon > 0$. It is well-known [33] that for $0 < p < q < \infty$, X is uniformly H_p-convex if and only if it is uniformly H_q-convex. Moreover we have

$$C_1 h_p^X(C_1 \epsilon^{q/p}) \leq h_q^X(\epsilon) \leq C_2 h_p^X(C_2 \epsilon) \quad (0 < \epsilon \leq 1),$$

where C_1, C_2 are two constants depending only on p, q and X. Thus we may say that X is **uniformly** H**-convex** if it is uniformly H_p-convex for some $0 < p < \infty$.

Given a Banach space X, recall that the **modulus of convexity** δ_X is defined by

$$\delta_X(\epsilon) = \inf\{1 - \|(x+y)/2\| : x, y \in B_X,\ \|x - y\| = \epsilon\},$$

for $0 \leq \epsilon \leq 2$, where B_X is the unit ball of the Banach space X, consisting of all elements $x \in X$ with $\|x\| \leq 1$. A Banach space X is said to be **uniformly**

convex if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$. We shall use the monotonicity property of δ_X [8, 26], that is, both $\epsilon \mapsto \delta_X(\epsilon)$ and $\epsilon \mapsto \delta_X(\epsilon)/\epsilon$ are increasing functions on $(0, 2]$.

Let us briefly sketch the contents of this paper. In Section 2, we study basic properties of moduli of monotonicity in quasi-Banach lattices. First we show that if a quasi-Banach lattice is uniformly monotone, then it does not contain any lattice isomorphic c_0-copy, so it is shown to be order continuous. Next we show that every uniformly monotone quasi-Banach space is α-convex for some $0 < \alpha < \infty$. In particular, it is shown that if a quasi-Banach space X is isomorphic to a uniformly monotone quasi-Banach lattice then ℓ_∞ can not be finitely λ-representable in X for any $\lambda \geq 1$.

In Section 3, we establish the relations between the uniform complex convexity and uniform monotonicity. More precisely, we show that if E is a complex quasi-Banach lattice with $M^{(\alpha)}(E) = 1$ for some $\alpha > 0$, then the following are equivalent (i) E is uniformly PL-convex; (ii) E is uniformly monotone; and (iii) E is uniformly $\mathbb{C}$-convex.

In Section 4, we investigate the relation between uniform monotonicity of a Banach lattice and uniform convexity of its convexification. First we show that if E is a uniformly monotone Banach lattice, then its p-convexification $E^{(p)}$ $(2 \leq p < \infty)$ is uniformly convex, extending analogous results for Köthe function spaces in [14]. Next we study basic quantitative properties of the modulus of monotonicity and moduli of convexity. Applying comparison of two moduli and Xu's idea in [34], we show that if E is a quasi-Banach lattice with $M^{(\alpha)}(\mathrm{E}) = 1$ for some $\alpha > 0$, then the following are equivalent: (i) E is uniformly H-convex of power type; (ii) E is uniformly PL-convex of power type; (iii) E is uniformly $\mathbb{C}$-convex of power type; and (iv) E is uniformly monotone of power type. We, in addition, get the Maurey–Pisier type theorem for a quasi-Banach lattice so that for any complex quasi-Banach lattice E the following properties are equivalent:

(1) E is q-concave for some $q < \infty$;
(2) E has a lattice renorming under which it is uniformly H-convex;
(3) E has a lattice renorming under which it is uniformly PL-convex;
(4) E has a lattice renorming under which it is uniformly $\mathbb{C}$-convex of power type;
(5) E has a lattice renorming under which it is uniformly monotone of power type;
(6) for any $\lambda \geq 1$, ℓ_∞ is not finitely λ-representable in E;
(7) for any $\lambda \geq 1$, ℓ_∞ is not lattice finitely λ-representable in E; and

(8) E has the super-$ARNP$;
We conclude this section with an extension of Theorem 4.4 in [34], so it is shown that if E is a σ-order continuous symmetric quasi-Banach function space on $(0,\infty)$ with $M^{\alpha}(E)=1$ and if E is uniformly monotone of power type, then $L_E(M,\tau)$ is uniformly H-convex for any semifinite von Neumann algebra (M,τ). For the definition of $L_E(M,\tau)$, see [34].

In the last section, we study the lifting property of uniform PL-convexity. Suppose that $(X,\|\cdot\|_X)$ is a continuously quasi-normed space and suppose also that E is a complex quasi-Köthe function space with $M^{(\alpha)}(E)=1$ for some $\alpha>0$. Then we show that the quasi-Köthe-Bochner function space $E(X)$ is uniformly PL-convex if and only if both E and X are uniformly PL-convex.

2. Modulus of monotonicity for quasi-Banach lattices

The modulus of monotonicity in Banach lattice has been introduced in [16] and [23]. Following these ideas we define the **modulus of p-monotonicity** Π_p^E, $0<p<\infty$, of a (real or complex) quasi-Banach lattice E as follows: for each $\epsilon\geq 0$,

$$\Pi_p^E(\epsilon)=\inf\{\|(|x|^p+|y|^p)^{1/p}\|-1 : x,y\in E,\ \|x\|=1,\|y\|\geq\epsilon\}.$$

It is clear that $\epsilon\mapsto\Pi_p^E(\epsilon)$ is increasing and $p\mapsto\Pi_p^E(\epsilon)$ is decreasing. It is also easy to see [23] that for each $\epsilon>0$,

$$\Pi_p^E(\epsilon)=\inf\{\|(|x|^p+|y|^p)^{1/p}\|-1 : x,y\in E \text{ and } \|x\|=1,\|y\|=\epsilon\}.$$

By the definition we have for every $p\geq 1$,

$$\Pi_p^E(\epsilon)\sim\Pi_1^{E^{(1/p)}}(\epsilon^p). \tag{2.1}$$

We say that a quasi-Banach lattice E is **uniformly p-monotone** if $\Pi_p^E(\epsilon)>0$ for all $\epsilon>0$. A quasi-Banach lattice E is said to be **uniformly monotone** if it is uniformly 1-monotone. By definition, a real quasi-Banach lattice is uniformly p-monotone if and only if its complexification is uniformly p-monotone. In particular, their moduli of p-monotonicity are the same.

We start with a generalization to quasi-Banach lattices of a result on a copy of c_0 in Banach lattices (cf. Theorem 1.a.5. in [26]). Recall that a real quasi-Banach lattice E is said to be **complete** (resp. **σ-complete**) if every order bounded set (resp. sequence) in E has a least upper bound.

Proposition 2.1: *A real quasi-Banach lattice E which is not σ-complete contains a lattice isomorphic c_0-copy.*

Proof: By the Aoki–Rolewicz theorem, we may assume that a quasi-norm of E is p-norm $0 < p \leq 1$. Let $\{x_n\}_{n=1}^{\infty} \subset E$ be an order bounded sequence which does not have a least upper bound. By replacing $\{x_n\}_{n=1}^{\infty}$ with the sequence $\{\bigvee_{j=1}^{n} x_j\}_{n=1}^{\infty}$ we can assume without loss of generality that $0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq x$, for some non-zero element x in E. Notice that the cone of positive elements is closed in E. So if $\{x_n\}_{n=1}^{\infty}$ converges in this norm to an element of E then this limit is also the least upper bound of $\{x_n\}_{n=1}^{\infty}$. This contradicts to the fact that $\{x_n\}_{n=1}^{\infty}$ has no least upper bound in E.

Hence there is an $\alpha > 0$ and a sequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ so that the vectors $u_j = x_{n_{j+1}} - x_{n_j}$ satisfy $\|u_j\|^p \geq \alpha$, $u_j \geq 0$ and $\sum_{k=1}^{j} u_k \leq x$ for all j.

We claim that for every $\epsilon > 0$ and every $\beta > 0$, there is a subsequence $\{v_k\}_{k=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ so that $\|(v_k - \beta v_1)^+\|^p \geq \alpha - \epsilon$ for all $k > 1$. Indeed, if this is not true then there is a subsequence $\{w_k\}_{k=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ such that $\|(w_k - \beta w_j)^+\|^p < \alpha - \epsilon$ for all $k > j$. It follows that for any k we have

$$\begin{aligned}\|x\|^p \geq \left\|\sum_{i=1}^{k} w_i\right\|^p &= \beta^{-p}\left\|k w_{k+1} - \sum_{i=1}^{k}(w_{k+1} - \beta w_i)\right\|^p \\ &= \beta^{-p}\left\|k w_{k+1} - \sum_{i=1}^{k}(w_{k+1} - \beta w_i)^+ + \sum_{i=1}^{k}(w_{k+1} - \beta w_i)^-\right\|^p.\end{aligned}$$

Since $k w_{k+1} \geq \sum_{i=1}^{k}(w_{k+1} - \beta w_i)^+$ we get the following

$$\|x\|^p \geq \beta^{-p}\left\|k w_{k+1} - \sum_{i=1}^{k}(w_{k+1} - \beta w_i)^+\right\|^p \geq \beta^{-p}(k\alpha - k(\alpha - \epsilon)) = \beta^{-p} k \epsilon$$

and this is a contradiction for large k.

Now fix $0 < \epsilon < \alpha/2$ and construct a sequence $\{v_k\}_{k=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ so that $\|(v_k - \beta v_1)^+\|^p \geq \alpha - \epsilon$ for all $k > 1$, where $\beta^p = 2\|x\|^p/\epsilon$. Put $y_1 = \beta^{-1}(\beta v_1 - x)^+$ and $y_k = (v_k - \beta v_1)^+$ for $k > 1$. It is clear that $y_1 \wedge y_k = 0$ for $k > 1$. By the choice of $\{v_k\}_{k=1}^{\infty}$ we also get $y_n \geq 0$, $\sum_{k=1}^{n} y_k \leq \sum_{k=1}^{n} v_k \leq x$ for all $n \geq 1$, $\|y_k\|^p \geq \alpha - \epsilon$ for $k > 1$ and

$$\|y_1\|^p = \|(v_1 - \beta^{-1}x)^+\|^p \geq \|v_1\|^p - \beta^{-p}\|x\|^p - \|(v_1 - \beta^{-1}x)^-\|^p \geq \alpha - \epsilon.$$

Applying this argument again to the sequence $\{y_k\}_{k=2}^{\infty}$, instead of $\{u_j\}_{j=1}^{\infty}$, and with $\epsilon/2$, instead of ϵ, we can produce a new sequence for which the norms of its elements are $\geq (\alpha - \epsilon - \epsilon/2)^{1/p}$, partial sum of elements is $\leq x$ and the first two elements are mutually disjoint and also disjoint from the rest of the sequence. Continuing by induction we obtain a sequence $\{z_k\}_{k=1}^{\infty}$, of mutually disjoint elements of E, so that $\|z_k\|^p \geq \alpha - 2\epsilon$ and $0 \leq z_k \leq x$ for all k. This sequence is clearly equivalent to the unit vector basis of c_0. ∎

A real quasi-Banach lattice E is said to be **order continuous** (resp. **σ-order continuous**) if for every decreasing net (resp. sequence) $\{x_\alpha\}_{\alpha\in A}$ in E with $\bigwedge_{\alpha\in A} x_\alpha = 0$, $\lim_\alpha \|x_\alpha\| = 0$.

In view of Proposition 2.1, the next two results can be proved analogously to Propositions 1.a.7 and 1.a.8 in [26].

PROPOSITION 2.2: *A σ-complete real quasi-Banach lattice E, which is not σ-order continuous, contains a subspace lattice isomorphic to ℓ_∞.*

PROPOSITION 2.3: *Let E be a real quasi-Banach lattice. Then the following assertions are equivalent:*

(1) *E is σ-complete and σ-order continuous;*
(2) *Every bounded increasing sequence in E converges in the quasi-norm topology of E;*
(3) *E is order continuous; and*
(4) *E is order continuous and order complete.*

We shall say that **a complex quasi-Banach lattice $E^{\mathbb{C}}$ is complete (**resp. **σ-complete, order continuous)** if E is complete (resp. σ-complete, order continuous). Then it is easy to see that Proposition 2.1, 2.2 and 2.3 hold for a complex quasi-Banach lattice.

PROPOSITION 2.4: *Let $1 \leq p < \infty$. Suppose that E is a uniformly p-monotone (real or complex) quasi-Banach lattice. Then E does not contain a lattice isomorphic c_0-copy. In particular, uniformly p-monotone quasi-Banach lattice is order continuous.*

Proof: Suppose, by contradiction, that E is uniformly p-monotone but that there is a lattice isomorphism $T : c_0 \to E$ such that there is a positive constant K with

$$K\|x\| \leq \|Tx\| \leq \|T\|\|x\|$$

for all $x \in c_0$. Then choose a sequence (x_n) in S_{c_0} with $\|Tx_n\| \geq (1/2)\|T\|$ such that $\lim_{n\to\infty} \|Tx_n\| = \|T\|$. Further we choose a sequence (y_n) in B_{c_0} with $\|y_n\|_{c_0} \geq 1/2$ so that $\||x_n| + |y_n|\|_{c_0} = 1$ for all $n \in \mathbb{N}$. Thus for every $n \in \mathbb{N}$,

$$\begin{aligned}\|Tx_n\|(1+\Pi_p^E(\|Ty_n\|/\|Tx_n\|) &\le \|(|Tx_n|^p+|Ty_n|^p)^{1/p}\| \\ &\le \|(|Tx_n|+|Ty_n|)\| \le \|T(|x_n|+|y_n|)\| \\ &\le \|T\|\||x_n|+|y_n|\|_{c_0} \le \|T\|.\end{aligned}$$

By taking the limit we obtain that

$$\lim_{n\to\infty} \Pi_p^E(\|Ty_n\|/\|Tx_n\|)=0.$$

Since $1/2 \le \|y_n\|_{c_0} \le K^{-1}\|Ty_n\|$, we have

$$\Pi_p^E(K/2\|T\|) \le \Pi_p^E(\|Ty_n\|/\|Tx_n\|)$$

for all $n \in \mathbb{N}$. This implies that $\Pi_p^E(K/2\|T\|)=0$, which is a contradiction to the fact that E is uniformly p-monotone. Then Proposition 2.1, 2.2 and 2.3 imply that E is order continuous. ∎

Lemma 2.5: *Suppose that E is an order continuous (real or complex) quasi-Banach lattice with $M^{(\alpha)}(E)=1$ for some $\alpha>0$. Let $p\ge 1$ and let x,y be non-zero positive elements in E. Then there are $\delta=\delta(\|x\|,\|y\|)>0$ and non-zero $z\in E^+$ such that $z\le y$, $\|z\|\ge\|y\|/2$ and*

$$(x^p+y^p)^{1/p} \ge x+\delta z.$$

In particular, we can take $\delta(\|x\|,\|y\|)=\frac{(2^p\|x\|^p+\|y\|^p)^{1/p}-2\|x\|}{\|y\|}$.

Proof: In the case of $\alpha\ge 1$, E is a Banach lattice and the result has already been shown in [23]. So assume that $0<\alpha<1$. Let G be an ideal of E with a weak unit such that $x,y\in G$. Following [34], since E is order continuous, G is order isomorphic to a quasi-Banach lattice of measurable functions on a probability space (Ω,Σ,μ) containing $L^\infty(\mu)$. Since $M^{(\alpha)}(E)=1$ holds, it follows that $G\hookrightarrow L^\alpha(\mu)$ (inclusion of norm 1). Therefore we can assume that E itself is a separable quasi-Banach lattice of measurable functions in (Ω,Σ,μ) such that

$$L^\infty(\mu)\hookrightarrow E\hookrightarrow L^\alpha(\mu) \quad \text{(inclusions of norm 1)}.$$

Let

$$A=\Big\{t\in\Omega : x(t)<\frac{k\|x\|}{\|y\|}y(t)\Big\}, \quad k=\Big(\frac{2^\alpha}{2^\alpha-1}\Big)^{1/\alpha},$$

we get

$$\|x\|\ge\|x\chi_{\Omega\setminus A}\|\ge\frac{k\|x\|}{\|y\|}\|y\chi_{\Omega\setminus A}\|.$$

Taking $z = y\chi_A$, $z \le y$ and

$$\|z\|^\alpha \ge \|y\|^\alpha - \|y\chi_{\Omega\setminus A}\|^\alpha \ge \left(\frac{\|y\|}{2}\right)^\alpha.$$

On the other hand, notice that for each $\epsilon > 0$ there is $\delta_1 = \delta_1(\epsilon) > 0$ such that for each $a \ge \epsilon$,

$$(1 + a^p)^{1/p} \ge 1 + \delta_1 a.$$

In fact, it is easy to check that we can take

$$\delta_1(\epsilon) = \frac{(1+\epsilon^p)^{1/p} - 1}{\epsilon}.$$

Hence if we take $\delta = \delta_1(\|y\|/\|2x\|)$ then

$$\begin{aligned}(x^p + y^p)^{1/p} &= (x^p\chi_A + y^p\chi_A)^{1/p} + (x^p\chi_{\Omega\setminus A} + y^p\chi_{\Omega\setminus A})^{1/p}\\ &\ge x\chi_A + \delta y\chi_A + x\chi_{\Omega\setminus A}\\ &= x + \delta z,\end{aligned}$$

and we obtain the desired result. ∎

By Proposition 2.4 and Lemma 2.5, we immediately obtain the following result.

PROPOSITION 2.6: *Suppose that E is a (real or complex) quasi-Banach lattice with $M^{(\alpha)}(E) = 1$ for some $\alpha > 0$. Then for each $1 \le p < \infty$, E is uniformly p-monotone if and only if E is uniformly monotone. In particular we obtain the following inequalities: for each $1 \le p < \infty$ and for each $\epsilon > 0$,*

$$\Pi_1^E(\epsilon^p) \preceq \Pi_p^E(\epsilon) \le \Pi_1^E(\epsilon).$$

Observe that a Banach lattice E is uniformly monotone with $\Pi_p^E \succeq \epsilon^r$ for some $1 \le p < \infty$ and for some $r \ge 1$ if and only if there is a $\lambda > 0$ such that

$$\|(|x|^p + |y|^p)^{1/p}\| \ge (\|x\|^r + \lambda\|y\|^r)^{1/r}$$

for all x and y in E. We shall denote the largest possible value of λ by $J_{r,p}(E)$. Then, by induction, it is clear that

$$\left\|\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\right\| \ge \left(\|x_1\|^r + J_{r,p}(E)\sum_{k=2}^n \|x_k\|^r\right)^{1/r} \tag{2.2}$$

for every $x_1, \ldots, x_n$ in E. This is an analogue of formula (1.3) concerning moduli of r-uniformly PL-convexity. We shall use this fact in the proofs of Propositions 2.8 and 4.3.

Recall that for $\lambda \geq 1$, ℓ_∞ is **lattice finitely λ-representable** in a real (resp. complex) quasi-Banach lattice E if given $\epsilon > 0$ and $n \in \mathbb{N}$ there exist $x_i \geq 0$ $(1 \leq i \leq n)$ so that $x_i \wedge x_j = 0$ $(i \neq j)$, $\|x_i\| \leq \lambda$ $(1 \leq i \leq n)$ and whenever $a_1, \ldots, a_n \in \mathbb{R}$(resp. $\mathbb{C}$), we have (cf. [21])

$$\max_{1 \leq i \leq n} |a_i| \leq \|a_1 x_1 + \cdots + a_n x_n\| \leq \lambda(1+\epsilon) \max_{1 \leq i \leq n} |a_i|.$$

Notice that ℓ_∞ cannot be a lattice finitely 1-represented in a uniformly monotone quasi-Banach lattice E. Then by Theorem 4.1 in [21], E is α-convex for some $0 < \alpha < \infty$. This proves the next proposition.

Proposition 2.7: *Suppose that E is a (real or complex) uniformly monotone quasi-Banach lattice. Then E is α-convex for some $0 < \alpha < \infty$.*

For $\lambda \geq 1$, ℓ_∞ is said to be **finitely λ-representable** in a real (resp. complex) quasi-Banach space X if for every $\epsilon > 0$ and every $n \in \mathbb{N}$ there exist $x_i \in X$ $(1 \leq i \leq n)$ so that whenever $a_1, \ldots, a_n \in \mathbb{R}$ (resp. $\mathbb{C}$), we have

$$\max_{1 \leq i \leq n} |a_i| \leq \|a_1 x_1 + \cdots + a_n x_n\| \leq \lambda(1+\epsilon) \max_{1 \leq i \leq n} |a_i|$$

(for more details, see [9]). Notice that if ℓ_∞ can not be finitely λ-representable in a quasi-Banach space X for every $\lambda \geq 1$, then X does not contain any subspace which is isomorphic to c_0.

In the case that a modulus of monotonicity is of power type, we obtain the next proposition (cf. Proposition 2.4).

Proposition 2.8: *Let X be a (real or complex) quasi-Banach space. Suppose that X is isomorphic to a quasi-Banach lattice E which is uniformly monotone of power type. Then ℓ_∞ cannot be finitely λ-representable in X for any $\lambda \geq 1$. In particular, X does not contain any subspace which is isomorphic to c_0.*

Proof: Notice that if a quasi-Banach space X_1 is isomorphic to a quasi-Banach space X_2 and if ℓ_∞ is finitely λ-representable in X_1 for some $\lambda \geq 1$ then ℓ_∞ is finitely λ'-representable in X_2 for some $\lambda' \geq 1$. So we have only to show that ℓ_∞ is not finitely λ-representable in E for any $\lambda \geq 1$.

Notice that by Proposition 2.7, E is α-convex for some $0 < \alpha < \infty$. Since the modulus of monotonicity Π_2^E is of power type ϵ^r, by equation (2.2), there is a positive constant $J > 0$ such that

$$\left\| \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} \right\| \geq \left(\|x_1\|^r + J \sum_{k=2}^{n} \|x_k\|^r \right)^{1/r} \tag{2.3}$$

for every $x_1, \dots, x_n$ in E.

Recall that by the Khinchin inequality (see [9]) and by the Krivine functional calculus, for any $0 < p < \infty$, there are constants A_p and B_p depending only on p such that for every finite sequence $x_1, \dots, x_n$ in E,

$$\begin{aligned} A_p \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\| &\le \left\| \left(\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left| \sum_{j=1}^n \epsilon_j x_j \right|^p \right)^{1/p} \right\| \\ &\le B_p \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\| \end{aligned} \tag{2.4}$$

where $\sum_{\epsilon_i = \pm 1}$ means the sum over all choices of $\epsilon_1, \dots, \epsilon_n = \pm 1$.

Suppose, on the contrary, that ℓ_∞ is finitely λ-representable in E for some $\lambda \ge 1$. So for every $n \in \mathbb{N}$, there exist $x_i \in E$ $(1 \le i \le n)$ so that whenever $a_1, \dots, a_n \in \mathbb{C}$, we have

$$\max_{1 \le i \le n} |a_i| \le \|a_1 x_1 + \cdots + a_n x_n\| \le 2\lambda \max_{1 \le i \le n} |a_i|.$$

Then

$$\begin{aligned} &\left\| \left(\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left| \sum_{j=1}^n \epsilon_j x_j \right|^\alpha \right)^{1/\alpha} \right\| \\ &\qquad \ge A_\alpha \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\| \ge A_\alpha \left(\|x_1\|^r + J \sum_{k=2}^n \|x_k\|^r \right)^{1/r} \ge A_\alpha J n^{1/r}. \end{aligned}$$

Hence we have for every n,

$$\begin{aligned} A_\alpha J n^{1/r} &\le \left\| \left(\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left| \sum_{j=1}^n \epsilon_j x_j \right|^\alpha \right)^{\frac{1}{\alpha}} \right\| \le M^{(\alpha)}(E) \left(\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^\alpha \right)^{\frac{1}{\alpha}} \\ &\le 2 M^{(\alpha)}(E) \lambda, \end{aligned}$$

which is a contradiction and this completes the proof. ∎

As an example, we shall compute the moduli of monotonicity of L^p.

Example 2.9: Let $0 < p, q < \infty$ and E be an L^p-space over a measure space (Ω, Σ, μ). Suppose that $0 < p \le q \le r < \infty$. Then the Minkowski inequality shows that for every $x, y \in E$,

$$\begin{aligned} \|(|x|^q + |y|^q)^{\frac{1}{q}}\|_p &= \left(\int_\Omega (|x(t)|^q + |y(t)|^q)^{p/q} dt \right)^{1/p} \\ &\ge (\|x\|_p^q + \|y\|_p^q)^{1/q} \ge (\|x\|_p^r + \|y\|_p^r)^{1/r}. \end{aligned}$$

Hence $\Pi_q^{L^p}(\epsilon) \succeq \epsilon^q$ and $J_{r,q}(L^p) = 1$ for all $0 < p \leq q \leq r < \infty$. Then $\Pi_2^{L^p}(\epsilon) \succeq \epsilon^2$ for all $0 < p \leq 2$ and $\Pi_2^{L^p}(\epsilon) \geq \Pi_p^{L^p}(\epsilon) \succeq \epsilon^p$ for all $p \geq 2$. ∎

3. Uniform monotonicity and uniform complex convexity in quasi-Banach Lattices

Let E be a complex quasi-Banach lattice. Then it is easy to see that for each $x, y \in E$ with $\|x\| = 1$, $\|y\| = \epsilon$,

$$\||x| + |y|\| \geq \sup\{\|x + \zeta y\| : |\zeta| \leq 1\} \geq 1 + H_\infty^E(\epsilon).$$

Hence $\Pi_1^E(\epsilon) \geq H_\infty^E(\epsilon)$ and the next result follows immediately.

Proposition 3.1: *Let E be a complex quasi-Banach lattice. Then for every $0 < p < \infty$ and every $\epsilon > 0$,*

$$\Pi_1^E(\epsilon) \geq H_\infty^E(\epsilon) \geq H_p^E(\epsilon).$$

Proposition 3.2: *Let E be a complex quasi-Banach lattice with $M^{(\alpha)}(E) = 1$ for some $\alpha > 0$. Then*

$$\Pi_1^E(\epsilon) \geq H_\infty^E(\epsilon) \geq H_{\min\{1,\alpha\}}^E(\epsilon) \geq \Pi_2^E\left(\epsilon\sqrt{I_{2,\alpha}(\mathbb{C})}\right).$$

Proof: For the case of $\alpha \geq 1$, E is a Banach lattice and this inequality was shown in [23]. So we may assume that $0 < \alpha < 1$. We have only to prove the third inequality. Note that we may assume that $\Pi_2^E(\epsilon) > 0$ for every $\epsilon > 0$. Using the same idea as in the proof of Lemma 2.5, we may assume that E is itself a separable quasi-Banach lattice of measurable functions on a probability measure space (Ω, Σ, μ) such that

$$L^\infty(\mu) \hookrightarrow E \hookrightarrow L^\alpha(\mu) \quad \text{(inclusions of norm 1)}.$$

By [7], $H_\alpha^{\mathbb{C}}$ is of power type ϵ^2. So there is a positive number $I = I_{2,\alpha}(\mathbb{C})$ such that for any complex numbers z_1, z_2,

$$\left(\int_{\mathbb{T}} |z_1 + e^{i\theta} z_2|^\alpha dm(\theta)\right)^{1/\alpha} \geq (|z_1|^2 + I|z_2|^2)^{1/2}.$$

Now, applying the Krivine functional calculus, we get, for any x, y in E,

$$\left(\int_{\mathbb{T}} |x + e^{i\theta} y|^\alpha dm(\theta)\right)^{1/\alpha} \geq (|x|^2 + I|y|^2)^{1/2}.$$

Consider the simple function on $[0, 2\pi]$

$$f(\theta) = \sum_{k=1}^{n} a_k \chi_{G_k}(\theta),$$

where G_k are mutually disjoint Lebesgue measurable subsets of $[0, 2\pi]$ and $a_k \in E$. Then the α-convexity of E with $M^{(\alpha)} = 1$ gives the following

$$\left\| \left(\sum_{i=1}^{n} |a_i|^\alpha m(G_i) \right)^{1/\alpha} \right\| \le \left(\sum_{i=1}^{n} \|a_i\|^\alpha m(G_i) \right)^{1/\alpha}.$$

Hence for every simple function $f : [0, 2\pi] \to E$,

$$\left\| \left(\int_{\mathbb{T}} |f|^\alpha dm \right)^{1/\alpha} \right\| \le \left(\int_{\mathbb{T}} \|f\|^\alpha dm \right)^{1/\alpha}. \tag{3.1}$$

Now we find a sequence of simple functions that approximate the element $|x + e^{i\theta} y|$. For each n, choose

$$a_k(t) = \inf \left\{ |x(t) + e^{i\theta} y(t)| : \theta \in \left[\frac{2\pi(k-1)}{2^n}, \frac{2\pi k}{2^n}\right),\ \theta \in \mathbb{Q} \right\},\ k = 1, \ldots, 2^n.$$

With

$$f_n(\theta, t) = \sum_{k=1}^{2^n} a_k(t) \chi_{\left[\frac{2\pi(k-1)}{2^n}, \frac{2\pi k}{2^n}\right)}(\theta), \quad \theta \in [0, 2\pi],$$

we obtain $0 \le f_n(\theta, t) \uparrow |x(t) + e^\theta y(t)|$ for every $\theta \in \mathbb{T}$ and for every $t \in \Omega$. Then applying the monotone convergence theorem, we have for each $t \in \Omega$

$$\int_{\mathbb{T}} f_n(\theta, t)^\alpha dm(\theta) \uparrow \int_{\mathbb{T}} |x(t) + e^\theta y(t)|^\alpha dm(\theta).$$

Using Proposition 2.3 and 2.4, we have

$$\lim_{n\to\infty} \|f_n(\theta, \cdot)\|^\alpha = \|x + e^{i\theta} y\|^\alpha$$

and

$$\lim_{n\to\infty} \left\| \left(\int_{\mathbb{T}} f_n(\theta, \cdot)^\alpha dm(\theta) \right)^{1/\alpha} \right\| = \left\| \left(\int_{\mathbb{T}} |x + e^{i\theta} y|^\alpha dm(\theta) \right)^{1/\alpha} \right\|.$$

Putting f_n instead of f in inequality (3.1) and taking a limit, we have, for each $x \in S_E$ and $y \in E$ with $\|y\| \ge \epsilon$,

$$\begin{aligned} \left(\int_{\mathbb{T}} \|x + e^{i\theta} y\|^\alpha dm \right)^{1/\alpha} &\ge \left\| \left(\int_{\mathbb{T}} |x + e^{i\theta} y|^\alpha dm \right)^{1/\alpha} \right\| \\ &\ge \left\| (|x|^2 + I|y|^2)^{1/2} \right\| \ge 1 + \Pi_2^E(\sqrt{I}\epsilon), \end{aligned}$$

and we obtain the desired result. ∎

It is shown in [10] that if a sequence $\{x_n\}$ is unconditionally summable in a quasi-Banach space X, then $\sum_n H_\infty^X(\|x_n\|) < \infty$. We obtain here the monotone version of this result by Proposition 3.2.

COROLLARY 3.3: *Suppose that E is a (real or complex) quasi-Banach lattice with $M^{(\alpha)}(E) = 1$. Then for every unconditionally summable sequence $\{x_n\}$ in E,*

$$\sum_n \Pi_2^E(\|x_n\|) < \infty.$$

Propositions 2.6 and 3.2 give the following theorem, which is a generalization of the corresponding results in [17, 23].

THEOREM 3.4: *Let E be a complex quasi-Banach lattice with $M^{(\alpha)}(E) = 1$ for $\alpha > 0$. Then the following properties are equivalent:*

(1) *E is uniformly PL-convex;*
(2) *E is uniformly monotone; and*
(3) *E is uniformly $\mathbb{C}$-convex.*

Recall that, in quasi-Banach lattices without the condition $M^{(\alpha)} = 1$, the uniform $\mathbb{C}$-convexity does not necessarily imply the uniform PL-convexity (see [24] (cf. [28])).

4. Uniform monotonicity and uniform convexity of its p-convexification in quasi-Banach lattices

We start with an auxiliary inequality for complex numbers. The analogous inequality for real numbers is well known (cf. Lemma 1.f.2 in [26]).

LEMMA 4.1: *Let $q \geq 2$; then for any $1 < p < \infty$, there exists a constant $C = C(p, q) > 0$ such that for every choice of complex numbers s and t,*

$$\Big(\Big|\frac{s-t}{C}\Big|^q + \Big|\frac{s+t}{2}\Big|^q\Big)^{1/q} \leq \Big(\frac{|s|^p + |t|^p}{2}\Big)^{1/p}. \tag{4.1}$$

Proof: Notice that the left (resp. right) side of inequality (4.1) is decreasing (resp. increasing) function of $q \geq 2$ (resp. $p > 1$) for fixed s, t in $\mathbb{C}$. Hence it suffices to prove the inequality for $q = 2$ and $1 < p < 2$. Notice also that

$$\begin{aligned}\Big|\frac{s-t}{C}\Big|^2 + \Big|\frac{s+t}{2}\Big|^2 &= \Big(\frac{1}{4} + \frac{1}{C^2}\Big)(|s|^2 + |t|^2) + \Big(\frac{1}{2} - \frac{2}{C^2}\Big)\mathrm{Re}(s\bar{t}) \\ &\leq \Big(\frac{1}{4} + \frac{1}{C^2}\Big)(|s|^2 + |t|^2) + \Big(\frac{1}{2} - \frac{2}{C^2}\Big)|s||t|.\end{aligned} \tag{4.2}$$

Thus, for $C > 2$, the expression (4.2) is maximized when s and t are positive real numbers. Therefore it follows from the real case, Lemma 1.f.2 in [26], that for every $1 < p < 2$, there is a constant $C > 2$ such that for any s, t in $\mathbb{C}$,

$$\Big(\Big|\frac{s-t}{C}\Big|^2 + \Big|\frac{s+t}{2}\Big|^2\Big)^{1/2} \le \Big(\frac{|s|^p + |t|^p}{2}\Big)^{1/p}.$$

Hence we obtained the desired result. ∎

The next three results are partial generalizations of Corollary 2 in [16] from Köthe function spaces to Banach lattices.

PROPOSITION 4.2: *Suppose that E is a uniformly monotone (real or complex) Banach lattice. Then $E^{(p)}$ is uniformly convex for $p \ge 2$ and*

$$\delta_{E^{(p)}}(\epsilon) \succeq \Pi_1^E(\epsilon^p).$$

Proof: We follow the notation of p-convexification of a Banach lattice in [26]. Let $(E^{(p)}, \oplus, \odot, \|\!| \cdot \|\!|)$ be a p-convexification of E.

Let x, y be elements of in the unit sphere of $E^{(p)}$ with

$$\|\!|x \oplus (-y)\|\!| = \|(x^{1/p} - y^{1/p})^p\|^{1/p} = \epsilon.$$

Then by Lemma 4.1 and the Krivine functional calculus, setting $p = q \ge 2$, we obtain

$$\begin{aligned} 1 &\ge \frac{\|x\| + \|y\|}{2} \ge \Big\|\frac{|x| + |y|}{2}\Big\| \ge \Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^p + \Big|\frac{x^{1/p} - y^{1/p}}{C}\Big|^p\Big\| \\ &\ge \Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^p\Big\|\Bigg[1 + \Pi_1^E\Bigg(\frac{\Big\|\Big|\frac{x^{1/p} - y^{1/p}}{2}\Big|^p\Big\|}{C^p\Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^p\Big\|}\Bigg)\Bigg] \\ &\ge \Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^p\Big\|\Big[1 + \Pi_1^E\Big(\frac{\epsilon^p}{(2C)^p}\Big)\Big]. \end{aligned}$$

Therefore

$$\|\!|(x \oplus y) \odot 2^{-1}\|\!| = \Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^p\Big\|^{1/p} \le \Bigg(\frac{1}{1 + \Pi_1^E\big(\frac{\epsilon^p}{(2C)^p}\big)}\Bigg)^{1/p},$$

and the proof is done. ∎

Notice in Proposition 4.2 that the modulus of convexity is of power type if Π_p^E is of power type. Moreover, in the case that the modulus of monotonicity Π_p^E is of power type, we obtain the stronger version of the previous result.

Proposition 4.3: *Suppose that E is a (real or complex) Banach lattice with $\Pi_2^E(\epsilon) \succeq \epsilon^r$. Then $E^{(p)}$ is uniformly convex with modulus of convexity $\delta_{E^{(p)}}(\epsilon) \succeq \epsilon^{pr}$ for all $p > 1$.*

Proof: We follow the notation of p-convexification of a Banach lattice in [26]. Let $(E^{(p)}, \oplus, \odot, \|\!| \cdot |\!\|)$ be a p-convexification of E.

Let x, y be elements in the unit sphere of $E^{(p)}$ with

$$|\!\|x \oplus (-y)|\!\| = \left\|(x^{1/p} - y^{1/p})^p\right\|^{1/p} = \epsilon.$$

For $p > 1$, taking $q = 2p$ in Lemma 4.1, we obtain, by the Krivine functional calculus,

$$\Big(\Big|\frac{x^{1/p} - y^{1/p}}{C}\Big|^{2p} + \Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^{2p}\Big)^{1/2} \leq \Big(\frac{|x| + |y|}{2}\Big).$$

Since the modulus of monotonicity is of power type ϵ^r, by the equation (2.2), there is a positive constant $J > 0$ such that

$$\left\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\right\| \geq \Big(\|x_1\|^r + J\sum_{k=2}^n \|x_k\|^r\Big)^{1/r}$$

for every $x_1, \ldots, x_n$ in E. This gives the following inequalities:

$$\begin{aligned} 1 \geq \frac{\|x\| + \|y\|}{2} \geq \Big\|\frac{|x| + |y|}{2}\Big\| &\geq \Big\|\Big(\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^{2p} + \Big|\frac{x^{1/p} - y^{1/p}}{C}\Big|^{2p}\Big)^{1/2}\Big\| \\ &\geq \Big(\Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^{p}\Big\|^r + J\Big\|\Big|\frac{x^{1/p} - y^{1/p}}{C}\Big|^{p}\Big\|^r\Big)^{1/r} \\ &\geq \Big(\Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^{p}\Big\|^r + J\Big(\frac{\epsilon}{C}\Big)^{pr}\Big)^{1/r}. \end{aligned}$$

Therefore,

$$|\!\|(x \oplus y) \odot 2^{-1}|\!\| = \Big\|\Big|\frac{x^{1/p} + y^{1/p}}{2}\Big|^{p}\Big\|^{1/p} \leq \Big(1 - J\Big(\frac{\epsilon}{C}\Big)^{pr}\Big)^{1/pr},$$

and this completes the proof. ∎

It is shown in [14] that if E is a uniformly convex Banach lattice, it is also uniformly monotone. In the following proposition, we refine this fact.

Proposition 4.4: *Suppose that a (real or complex) Banach lattice E is uniformly convex. Then it is uniformly monotone. In particular, we get for every $0 < \epsilon < 1$,*

$$\Pi_1^E(\epsilon) \geq 2\delta_E(\epsilon).$$

Proof: Let $0 < \epsilon \leq 1$ and let $\|x\| = 1$ and $\|y\| = \epsilon$. Consider the following vectors in E

$$S = |x| + |y|, \ a = \frac{S}{\|S\|} \quad \text{and} \quad b = \frac{S - 2|y|}{\|S\|}.$$

Then $a, b \in B_E$ and so

$$\left\|\frac{a+b}{2}\right\| \leq 1 - \delta_E(\|a-b\|).$$

Thus

$$\frac{\|x\|}{\|S\|} \leq 1 - \delta_E\Big(\frac{2\|y\|}{\|S\|}\Big),$$

that is

$$\delta_E\Big(\frac{2\|y\|}{\|S\|}\Big) \leq 1 - \frac{\|x\|}{\|S\|} = \frac{\|S\| - \|x\|}{\|S\|}.$$

Notice that $2/\|S\| \geq 1$ so that by the monotonicity of $\delta_E(\epsilon)/\epsilon$,

$$\frac{\delta_E(\|y\|)}{\|y\|} \leq \frac{\delta_E(\frac{2}{\|S\|}\|y\|)}{\frac{2}{\|S\|}\|y\|}.$$

It follows that

$$2\delta_E(\|y\|) \leq \|S\| \cdot \delta_E\Big(\frac{2\|y\|}{\|S\|}\Big) \leq \|S\| - \|x\|.$$

Therefore

$$1 + 2\delta_E(\epsilon) \leq \||x| + |y|\|,$$

and we obtain the desired result. ∎

The following is a partial generalization of results in [14]. It should be recalled that the modulus of convexity always satisfy (see [27])

$$\limsup_{\epsilon \to 0} \delta_X(\epsilon)/\epsilon^2 < \infty.$$

Therefore the modulus of convexity of power type should be of the form ϵ^r, where $r \geq 2$.

Corollary 4.5: *Let E be a (real or complex) quasi-Banach lattice with $M^{(\alpha)}(E) = 1$. Then the following are equivalent:*

(1) *E is uniformly monotone.*
(2) *$E^{(p)}$ is uniformly monotone for every $0 < p < \infty$.*
(3) *$E^{(p)}$ is uniformly monotone for some $0 < p < \infty$.*
(4) *$E^{(p/\alpha)}$ is uniformly convex for every $2 \leq p < \infty$.*
(5) *$E^{(p/\alpha)}$ is uniformly convex for some $2 \leq p < \infty$.*

Proof: We prove the implication (1) $\Rightarrow$ (2). Suppose that E is uniformly monotone. Then equation (2.1) and Proposition 2.6 imply that $E^{(p)}$ is uniformly monotone for $0 < p \leq 1$. Notice that for $0 < q < 1$ we have for every x and y in E,

$$\|(|x|^q + |y|^q)^{1/q}\| \geq \||x| + |y|\|.$$

Hence by the definition of modulus of monotonicity and q-convexification, we have for every $0 < q < 1$

$$\Pi_1^{E^{(1/q)}}(\epsilon^q) \succeq \Pi_1^E(\epsilon).$$

Hence $E^{(p)}$ is uniformly monotone for $p > 1$. The implication (1) $\Rightarrow$ (2) is proved.

The implication (2) $\Rightarrow$ (3) is trivial. For the implication (3) $\Rightarrow$ (4), assume that $E^{(p)}$ is uniformly monotone for some $0 < p < \infty$ and notice that $E^{(1/\alpha)}$ is a Banach lattice and it is uniformly monotone by the implication (1) $\Rightarrow$ (2). Then Proposition 4.2 implies that $E^{(p/\alpha)}$ is uniformly convex for $p \geq 2$.

The implication (4) $\Rightarrow$ (5) is trivial. Finally, suppose that (5) holds. Notice that $E^{(p/\alpha)}$ is a Banach lattice. Then Proposition 4.4 shows that $E^{(p/\alpha)}$ is uniformly monotone. Then the implication (1) $\Rightarrow$ (2) shows that E is uniformly monotone and (5)$\Rightarrow$(1) is proved. ∎

If we use Proposition 4.3 instead of Proposition 4.2 in the proof of Corollary 4.5, we have the following

Corollary 4.6: *Let E be a (real or complex) quasi-Banach lattice with $M^{(\alpha)}(E) = 1$. Then the following are equivalent:*

(1) *E is uniformly monotone of power type;*
(2) *$E^{(p)}$ is uniformly monotone of power type for every $0 < p < \infty$;*
(3) *$E^{(p)}$ is uniformly monotone of power type for some $0 < p < \infty$;*
(4) *$E^{(p/\alpha)}$ is uniformly convex of power type for every $1 < p < \infty$;*
(5) *$E^{(p/\alpha)}$ is uniformly convex of power type for some $1 < p < \infty$;*

In the next theorem we follow the outline of the proof of Theorem 3.2 in [34] applying Proposition 4.3 instead of the fact that E is uniformly convex of power type if $M^{(p)}(E) = M_{(q)}(E) = 1$ for some $1 < p, q < \infty$.

Theorem 4.7: *Let $0 < \alpha \leq q < \infty$. Let E be a quasi-Banach lattice with $M^{(\alpha)}(E) = 1$. Suppose that the modulus of monotonicity Π_2^E is of power type ϵ^q. Then E is uniformly H-convex. More precisely, for any $f \in H^p(E)$, where $p = \max\{2, q(1 + [1/\min\{\alpha, 1\}])\}$, we have*

$$(\|f(0)\|_E^p + \delta\|f - f(0)\|_{H^p(E)}^p)^{1/p} \leq \|f\|_{H^p(E)},$$

where $\delta > 0$ is a constant depending only on α and q. Consequently, $h_p^E(\epsilon) \succeq \epsilon^p$.

Proof: Notice that if $\alpha > 1$ then E is a Banach lattice and $M^{(1)}(E) = 1$ holds. Hence we may assume that $0 < \alpha \leq 1$. Let $n = 1 + [1/\alpha]$, where $[a]$ is the largest integer less than or equal to the real number a. Then $M^{(\alpha)}(E) = 1$ implies that $M^{(\alpha n)}(E^{(n)}) = 1$ and $E^{(n)}$ is a Banach lattice. Since $\alpha n > 1$, by Proposition 4.3 $E^{(n)}$ is uniformly convex of power type ϵ^p, where $p = \max(2, nq)$. Now let $f \in H^p(E^{(n)})$. Noting that $(f(0), f - f(0))$ is a martingale difference (for the definition of a martingale, see [29]) with values in $E^{(n)}$ and using Proposition 2.4 in [29] we have

$$(\|f(0)\|_{E^{(n)}}^p + \delta_0 \|f - f(0)\|_{H^p(E^{(n)})}^p)^{1/p} \leq \|f\|_{H^p(E^{(n)})},$$

where δ_0 is a constant depending only on α and q.

By Proposition 2.8, E does not have any c_0 isomorphic copy. Then we may use Theorem 3.1 in [34] so that we get n functions $f_1, \ldots, f_n$ with values in $E^{(n)}$ such that

$$f(z) = \prod_{k=1}^{n} f_k(z), \quad \text{for every } z \in \mathbb{D}$$

and

$$\|f_1\|_{H^p(E^{(n)})} = \|f\|_{H^p(E)}, \quad \|f_k\|_{H^\infty(E^{(n)})} = 1, \quad \text{for } 2 \leq k \leq n.$$

$M^{(\alpha)}(E) = 1$ implies that for x, y in E,

$$\|x + y\|_E^\alpha \leq \|x\|_E^\alpha + \|y\|_E^\alpha.$$

Applying the Hölder type inequality to the Banach lattice $E^{(1/\alpha)}$ we obtain (see Section 3 in [34]) that for any $0 < q_0, q_1, q < \infty$ with $1/q = 1/q_0 + 1/q_1$ and for any $x \in E^{(q_0)}$ and $y \in E^{(q_1)}$,

$$\|xy\|_{E^{(q)}} \leq \|x\|_{E^{(q_0)}} \|y\|_{E^{(q_1)}}.$$

Thus we have

$$\begin{aligned}
&\|f - f(0)\|_{H^p(E)}^p \\
&\leq \left\| \prod_{k=1}^{n} f_k - \prod_{k=1}^{n} f_k(0) \right\|_{H^p(E)}^p = \left\| \sum_{k=1}^{n} \prod_{j=1}^{k-1} f_j(0)(f_k - f_k(0)) \prod_{i=k+1}^{n} f_i \right\|_{H^p(E)}^p \\
&\leq \sup_{0 \leq r < 1} \int_{\mathbb{T}} \left[\sum_{k=1}^{n} \left\| \prod_{j=1}^{k-1} f_j(0)(f_k - f_k(0)) \prod_{i=k+1}^{n} f_i \right\|_E^\alpha \right]^{p/\alpha} dm \\
&\leq n^{\frac{p}{\alpha} - 1} \sum_{k=1}^{n} c_k,
\end{aligned}$$

where

$$c_k = \prod_{j=1}^{k-1} \|f_j(0)\|_{E^{(n)}}^p \|(f_k - f_k(0))\|_{H^p(E^{(n)})}^p \prod_{i=k+1}^{n} \|f_i\|_{H^\infty(E^{(n)})}^p.$$

Letting $\delta = \delta_0 n^{1-p/\alpha}$ and applying the Hölder type inequality again,

$$\begin{aligned}
&\|f(0)\|_E^p + \delta \|f - f(0)\|_{H^p(E)}^p \\
&\leq \Big[\prod_{k=1}^{n} \|f_k(0)\|_{E^{(n)}}\Big] + \delta_0 \sum_{k=1}^{n} c_k \\
&= \Big[\prod_{k=1}^{n-1} \|f_k(0)\|_{E^{(n)}}\Big] (\|f_n(0)\|_{E^{(n)}}^p + \delta_0 \|f_n - f_n(0)\|_{H^p(E^{(n)})}^p) + \delta_0 \sum_{k=1}^{n-1} c_k \\
&\leq \Big[\prod_{k=1}^{n-1} \|f_k(0)\|_{E^{(n)}}\Big] \|f_n\|_{H^p(E^{(n)})}^p + \delta_0 \sum_{k=1}^{n-1} c_k \\
&\leq \Big[\prod_{k=1}^{n-1} \|f_k(0)\|_{E^{(n)}}\Big] + \delta_0 \sum_{k=1}^{n-1} c_k \\
&\leq \cdots \leq \|f_1(0)\|_{E^{(n)}}^p + \delta_0 \|f_1 - f_1(0)\|_{H^p(E^{(n)})}^p \\
&\leq \|f_1\|_{H^p(E^{(n)})}^p = \|f\|_{H^p(E)}^p.
\end{aligned}$$

This proves the theorem with $\delta = \delta_0 n^{1-p/\alpha}$. ∎

The next theorem is a consequence of Proposition 2.6, 3.2 and Theorem 4.7.

Theorem 4.8: *Suppose that E is a quasi-Banach lattice with $M^{(\alpha)}(E)=1$ for some $\alpha > 0$. Then the following are equivalent:*

(1) *E is uniformly H-convex of power type;*
(2) *E is uniformly PL-convex of power type;*
(3) *E is uniformly $\mathbb{C}$-convex of power type; and*
(4) *E is uniformly monotone of power type.*

Notice that if E is q-concave, ℓ_∞ is not lattice finitely 1-representable in E, so by Theorem 4.1 in [21], E is α-convex for some $\alpha > 0$. Then, using the similar idea as in [34], we can obtain the following theorem (cf. Corollary 3.3 in [34], Corollary 4.4 and 4.6 in [23]). For the relevant facts concerning the super-properties, see [29].

Theorem 4.9: *For any quasi-Banach lattice $(E, \|\cdot\|)$ the following are equivalent:*

(1) *E is q-concave for some $q < \infty$;*

(2) *E has a lattice renorming under which it is uniformly H-convex;*
(3) *E has a lattice renorming under which it is uniformly PL-convex;*
(4) *E has a lattice renorming under which it is uniformly $\mathbb{C}$-convex of power type;*
(5) *E has a lattice renorming under which it is uniformly monotone of power type;*
(6) *for any $\lambda \geq 1$, ℓ_∞ is not finitely λ-representable in E;*
(7) *for any $\lambda \geq 1$, ℓ_∞ is not lattice finitely λ-representable in E; and*
(8) *E has the super-$ARNP$;*

Proof: The equivalence of (1), (2), (3) and (8) is shown in [34]. Suppose that (1) holds. E is α-convex and q-concave for some $\alpha > 0$. If we use the convexification and Proposition 1.d.8 in [26], we obtain a lattice renorming $|||\cdot|||$ with $M^{(\alpha)}(E) = M_{(q)}(E) = 1$. Then $(E, |||\cdot|||)$ is uniformly monotone of power type by Corollary 4.6 and it is uniformly $\mathbb{C}$-convex of power type by Theorem 4.8. Hence (1) implies (4). Proposition 3.1 shows that (4) implies (5). The implication (5)⇒(6) is proved by Proposition 2.8. (6)⇒(7) is trivial. We finish the proof by showing that (7) implies (1). Assume that (7) holds. By Theorem 4.1 in [21], E is α-convex for some $0 < \alpha < \infty$. Suppose, on the contrary, that E is not q-concave for any $q < \infty$. E admits an equivalent α-convex quasi-norm $|||\cdot|||$ with $M^{(\alpha)}(E) = 1$. Then the Banach lattice $F = E^{(1/\alpha)}$ is not q-concave for any $q < \infty$. Hence by Theorem 1.f.12 in [26], for any $\epsilon > 0$ and $n \in \mathbb{N}$ there exist mutually disjoint elements $x_i \geq 0$ $(1 \leq i \leq n)$ in F such that for all complex numbers a_i $(1 \leq i \leq n)$ we have

$$\max_{1\leq i\leq n} |a_i| \leq \|a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n\|_F \leq (1+\epsilon) \max_{1\leq i\leq n} |a_i|.$$

Using the definition of convexification and disjointness of x_i's, we get

$$\max_{1\leq i\leq n} |a_i| \leq |||a_1 x_1 + \cdots + a_n x_n|||_E \leq (1+\epsilon)^{1/\alpha} \max_{1\leq i\leq n} |a_i|.$$

Then ℓ_∞ is lattice finitely λ-representable in $(E, \|\cdot\|)$ for some λ. This is a contradiction to our assumption. ∎

Remark 4.10: In the proof of Theorem 4.9, it is easy to check that the equivalence of (1), (5), (6) and (7) can be established in real or complex quasi-Banach lattices, which the Maurey-Pisier type theorem (see [9]) for quasi-Banach lattices.

Let E be a σ-order continuous symmetric quasi-Banach function space on $(0, \infty)$. Hereafter, (M, τ) denotes a semifinite von Neumann algebra on a Hilbert space H, with a faithful semifinite normal trace τ and $L_E(M, \tau)$ the associated symmetric space of measurable operators. For the definition of the symmetric space $L_E(M, \tau)$ of measurable operators, consult [34].

If we review the proof of Theorem 4.4 in [34] with using Proposition 4.3 instead of the condition $M_q(E) = 1$, then it can be extended to the following

THEOREM 4.11: *Let $0 < \alpha \le q < \infty$. Let E be a complex symmetric quasi-Banach function space on $(0, \infty)$ with $M^{(\alpha)}(E) = 1$. Suppose that the modulus of monotonicity Π_2^E is of power type ϵ^q. Then $L_E(M, \tau)$ is uniformly H-convex for any semifinite von Neumann algebra (M, τ). More precisely, for any $f \in H^p(L_E(M, \tau))$, where $p = \max\{2, q(1 + [1/\min\{\alpha, 1\}])\}$, we have*

$$(\|f(0)\|_E^p + \delta\|f - f(0)\|_{H^p(L_E(E,\tau))}^p)^{1/p} \le \|f\|_{H^p(L_E(E,\tau))},$$

where $\delta > 0$ is a constant depending only on α and q. Consequently,

$$h_p^{L_E(E,\tau)}(\epsilon) \succeq \epsilon^p.$$

5. Lifting properties of uniform PL-convexity

Let E be a non-trivial quasi-Köthe function space over a complete measure space (Ω, μ). For the definition of a quasi-Köthe function space, see [22]. Let X be a non-trivial complex quasi-Banach space.

Let $L^0(X)$ be the set of all X-valued strongly μ-measurable functions. The **quasi-Köthe-Bochner function space** (cf. [25]) $E(X)$ is a quasi-Banach space defined by

$$E(X) = \{f \in L^0(X) \colon t \mapsto \|f(t)\|_X \text{ is an element of } E\},$$

with the quasi-norm

$$\|f\|_{E(X)} = \|\|f(\cdot)\|_X\|_E.$$

We show that a quasi-Köthe-Bochner function space $(E(X), \|\cdot\|_{E(X)})$ is a complete metric space and the quasi-norm is continuous if X is a continuously quasi-normed space.

PROPOSITION 5.1: *Let X be a quasi-Banach space and let E be an α-convex quasi-Köthe function space on a measure space (Ω, Σ, μ) for some $0 < \alpha < \infty$. Then $(E(X), \|\cdot\|_{E(X)})$ is a complete metric space.*

Proof: We have for every finite sequence $g_1, \ldots, g_n$ in $E(X)$ and $t \in \Omega$,

$$\Big\| \sum_{j=1}^{n} g_j(t) \Big\|_X \le B \Big(\sum_{j=1}^{n} \|g_j(t)\|_X^p \Big)^{1/p},$$

since X is p-normable for some $0 < p \le 1$ and $B > 0$. Let $\beta = \min\{p, \alpha\}$, then

$$\sum_{j=1}^{\infty} \|g_j\|_{E(X)}^{\beta} < \infty$$

for some sequence $\{g_j\}$ in $E(X)$. After renorming and convexification, we can apply Proposition 1.d.5 of [26] and we conclude that E is q-convex for every $0 < q < \alpha$. Hence

$$\Big\| \Big(\sum_{j=1}^{n} \|g_j(\cdot)\|_X^p \Big)^{1/p} \Big\|_E \le M^{(\beta)}(E) \Big(\sum_{j=1}^{n} \|g_j\|_{E(X)}^{\beta} \Big)^{1/\beta}.$$

This implies that

$$\Big\| \bigoplus_{i=1}^{n} \|g_j(\cdot)\|_X \Big\|_{E^{(1/p)}}^p = \Big\| \Big(\sum_{j=1}^{n} \|g_j(\cdot)\|_X^p \Big)^{1/p} \Big\|_E \le M^{(\beta)}(E) \Big(\sum_{j=1}^{n} \|g_j\|_{E(X)}^{\beta} \Big)^{1/\beta}.$$

By [6], $E^{(1/p)}$ is complete. So $\sum_{j=1}^{\infty} \|g_j\|_{E(X)}^{\beta} < \infty$ implies that

$$h(t) = \bigoplus_{j=1}^{\infty} \|g_j(t)\|_X$$

converges in $E^{(1/p)}$. Notice that $\bigoplus_{j=1}^{n} \|g_j(t)\|_X$ is increasing for $n \ge 1$. So it is easy to check that for almost every $t \in \Omega$,

$$h(t) = \bigoplus_{j=1}^{\infty} \|g_j(t)\|_X.$$

We also have for every $n \ge 1$ and $t \in \Omega$,

$$\Big\| \sum_{j=1}^{n} g_j(t) \Big\|_X \le B \Big(\sum_{j=1}^{n} \|g_j(t)\|_X^p \Big)^{1/p} \le B \bigoplus_{j=1}^{\infty} \|g_j(t)\|_X.$$

Hence it is shown that for almost every $t \in \Omega$,

$$g(t) = \sum_{j=1}^{\infty} g_j(t)$$

exists in X and $g \in E(X)$. For every $n \geq 1$

$$\Big\| g - \sum_{j=1}^{n} g_j \Big\|_{E(X)} \leq B \Big\| \Big(\sum_{j=n+1}^{\infty} \|g_j(\cdot)\|_X^p \Big)^{1/p} \Big\|_E$$
$$\leq M^{(\beta)}(E) B \Big(\sum_{j=n+1}^{\infty} \|g_j\|_{E(X)}^{\beta} \Big)^{1/\beta}.$$

Therefore $\{\sum_{j=1}^n g_j\}_n$ converges to g in $E(X)$ if $\sum_1^\infty \|g_j\|_{E(X)}^\beta$ is finite. The proof is complete. ∎

Proposition 5.2: *Suppose that $(X, \|\cdot\|_X)$ is a continuously quasi-normed space and suppose also that E is a complex quasi-Köthe function space with $M^{(\alpha)}(E) = 1$ for some $\alpha > 0$. Then the quasi-Köthe–Bochner function space $(E(X), \|\cdot\|_{E(X)})$ is a continuously quasi-normed space.*

Proof: We may assume that $0 < \alpha \leq 1$. Since $\|\cdot\|_X$ is uniformly continuous on the unit ball of X, given $\epsilon > 0$ there exists $\delta > 0$ such that $\big| \|x\|_X - \|y\|_X \big| < \epsilon$ if x and y are elements in B_X with $\|x - y\|_X \leq \delta$. Choose $\eta > 0$ so that $K(1 + 1/\delta)\eta < \epsilon$, where K is the quasi-norm constant of X.

Then if $\|f\|_{E(X)} \leq 1$ and $\|g\|_{E(X)} \leq 1$ and $\|f - g\|_{E(X)} \leq \eta$, let

$$A_1 = \{t \in \Omega : \|f(t) - g(t)\|_X < \delta \|g(t)\|_X \leq \delta \|f(t)\|_X\};$$
$$A_2 = \{t \in \Omega : \|f(t) - g(t)\|_X < \delta \|f(t)\|_X \leq \delta \|g(t)\|_X\};$$
$$B_1 = \{t \in \Omega : \|f(t) - g(t)\|_X \geq \delta \|g(t)\|_X\}; \quad \text{and}$$
$$B_2 = \{t \in \Omega : \|f(t) - g(t)\|_X \geq \delta \|f(t)\|_X\}.$$

If $t \in A_1$, let

$$f_1(t) = f(t)/\|f(t)\|_X, \ g_1(t) = g(t)/\|f(t)\|_X.$$

Then $\|f_1(t)\|_X = 1$, $\|g(t)\|_X \leq 1$, and $\|f_1(t) - g_1(t)\|_X \leq \delta$. Therefore,

$$\|f(t)\|_X - \|g(t)\|_X = \|f(t)\|_X (\|f_1(t)\| - \|g_1(t)\|_X) \leq \epsilon \|f(t)\|.$$

Note also that if $t \in B_1$,

$$\|f(t)\|_X = \|(f(t) - g(t)) + g(t)\|_X \leq K(\|f(t) - g(t)\|_X + \|g(t)\|_X)$$
$$\leq K(1 + 1/\delta)\|f(t) - g(t)\|_X.$$

Hence for every $t \in \Omega$,

$$\begin{aligned} &\|f(t)\|_X - \|g(t)\|_X \\ &\leq (\|f(t)\|_X - \|g(t)\|_X)\chi_{A_1}(t) + \|f(t)\|_X \chi_{B_1}(t) \\ &\leq \epsilon \|f(t)\|_X \chi_{A_1}(t) + K(1 + 1/\delta)\|f(t) - g(t)\|_X \chi_{B_1}(t) \end{aligned} \tag{5.1}$$

If we change the role of f and g in the inequality (5.1) we have

$$\|g(t)\|_X - \|f(t)\|_X \le \epsilon \|f(t)\|_X \chi_{A_2}(t) + K(1+1/\delta)\|f(t)-g(t)\|_X \chi_{B_2}(t)$$

So we have for every $t \in \Omega$,

$$\begin{aligned} |\|f(t)\|_X - \|g(t)\|_X| \le & \epsilon \|f(t)\|_X \chi_{A_1}(t) + K(1+1/\delta)\|f(t)-g(t)\|_X \chi_{B_1}(t) \\ & + \epsilon \|f(t)\|_X \chi_{A_2}(t) + K(1+1/\delta)\|f(t)-g(t)\|_X \chi_{B_2}(t). \end{aligned}$$

Hence we have

$$\begin{aligned} \|\|f(\cdot)\|_X - \|g(\cdot)\|_X\|_E^\alpha &\le 2\epsilon^\alpha \|f\|_{E(X)}^\alpha + 2K^\alpha (1+1/\delta)^\alpha \|f-g\|_{E(X)}^\alpha \\ &\le 2\epsilon^\alpha + 2K^\alpha(1+1/\delta)^\alpha \eta^\alpha \le 4\epsilon^\alpha. \end{aligned}$$

Therefore

$$|\|f\|_{E(X)}^\alpha - \|g\|_{E(X)}^\alpha| \le \|\|f(\cdot)\|_X - \|g(\cdot)\|_X\|_E^\alpha \le 4\epsilon^\alpha$$

This shows that $\|\cdot\|_{E(X)}$ is uniformly continuous on the unit ball of $E(X)$. ∎

Notice that if we choose $g \in E$ and $a \in X$ such that $\|g\|_E = 1$ and $\|a\|_X = 1$, then both, the map $x \mapsto g(\cdot)x$ from X into $E(X)$ and the map $f \mapsto f(\cdot)a$ from E into $E(X)$ are isometries.

The next result is a generalization of Theorem 5.2 in [23] from Banach to quasi-Banach spaces.

THEOREM 5.3: *Suppose that $(X, \|\cdot\|_X)$ is a continuously quasi-normed space and suppose also that E ia a complex quasi-Köthe function space with $M^{(\alpha)}(E) = 1$ for some $\alpha > 0$. Then the quasi-Köthe-Bochner function space $E(X)$ is uniformly PL-convex if and only if E is uniformly PL-convex and X is uniformly PL-convex.*

Proof: For $\alpha > 1$, E is a Banach lattice, so we have $M^{(1)}(E) = 1$. Hence we may assume that $0 < \alpha \le 1$. Suppose that $E(X)$ is uniformly PL-convex and suppose on the contrary, that E is not uniformly PL-convex. So there are sequences (x_n), (y_n) in E and $\epsilon > 0$ such that

$$\|x_n\|_E = 1, \quad \|y_n\|_E \ge \epsilon, \quad \text{and} \quad \lim_n \int_{\mathbb{T}} \|x_n + e^{i\theta} y_n\|_E \, dm(\theta) = 1.$$

Let a be a norm one element of X. Since $E(X)$ is uniformly PL-convex,

$$1 \le \int_{\mathbb{T}} \|x_n \otimes a + e^{i\theta} y_n \otimes a\|_{E(X)} \, dm(\theta) = \int_{\mathbb{T}} \|x_n + e^{i\theta} y_n\|_E \, dm(\theta)$$

holds for all $n \in \mathbb{N}$, where $x \otimes a$ is a function $\omega \mapsto x(\omega)a$ from Ω to X for every $x \in E$ and for every $a \in X$. Notice that $\|x_n \otimes a\|_{E(X)} = 1$ and $\|y_n \otimes a\|_{E(X)} \geq \epsilon$. Hence

$$\lim_n \int_{\mathbb{T}} \|x_n \otimes a + e^{i\theta} y_n \otimes a\|_{E(X)} \, dm(\theta) = 1.$$

This contradicts to the fact that $E(X)$ is uniformly PL-convex. By the isometric embedding of X into $E(X)$, X is uniformly PL-convex if $E(X)$ is uniformly PL-convex.

For the converse, suppose that both E and X are uniformly PL-convex. Consider the simple function

$$f(\theta) = \sum_{k=1}^{n} a_k \chi_{G_k}(\theta), \quad \theta \in [0, 2\pi],$$

where G_k are mutually disjoint Lebesgue measurable subsets of $\mathbb{T} = [0, 2\pi]$ and $a_k \in E$. Then the α-convexity of E with $M^{(\alpha)} = 1$ gives the following:

$$\left\| \left(\sum_{i=1}^{n} |a_i|^{\alpha} m(G_i) \right)^{1/\alpha} \right\|_E \leq \left(\sum_{i=1}^{n} \|a_i\|_E^{\alpha} m(G_i) \right)^{1/\alpha},$$

where $dm(t) = \frac{1}{2\pi} dt$ is the normalized Lebesgue measure on $\mathbb{T}$. Hence for every simple function f: $[0, 2\pi] \to E$,

$$\left\| \left(\int_{\mathbb{T}} |f|^{\alpha} \, dm \right)^{1/\alpha} \right\|_E \leq \left(\int_{\mathbb{T}} \|f\|_E^{\alpha} \, dm \right)^{1/\alpha} \tag{5.2}$$

holds.

Let x, y be elements in $E(X)$. Now we shall find simple functions that approximate $\|x + e^{i\theta} y\|_X$. For each n, let

$$a_k(t) = \inf \left\{ \|x(t) + e^{\theta} y(t)\|_X : \theta \in \left[\frac{2\pi(k-1)}{2^n}, \frac{2\pi k}{2^n} \right), \ \theta \in \mathbb{Q} \right\}, \ k = 1, \ldots, 2^n.$$

Letting

$$f_n(\theta, t) = \sum_{k=1}^{2^n} a_k(t) \chi_{[(2\pi(k-1))/2^n, (2\pi k)/2^n)}(\theta),$$

we obtain the simple functions f_n such that $0 \leq f_n(\theta, t) \uparrow \|x(t) + e^{i\theta} y(t)\|_X$ for every $t \in \Omega$ and for every $\theta \in \mathbb{T}$. Then applying the monotone convergence theorem, we have for each $t \in \Omega$

$$\int_{\mathbb{T}} f_n(\theta, t)^{\alpha} \, dm(\theta) \uparrow \int_{\mathbb{T}} \|x(t) + e^{\theta} y(t)\|_X^{\alpha} \, dm(\theta).$$

Using Proposition 2.3, we have for every $\theta \in \mathbb{T}$,

$$\lim_{n\to\infty} \|f_n(\theta,\cdot)\|_{E(X)}^\alpha = \|x + e^{i\theta} y\|_{E(X)}^\alpha$$

and

$$\lim_{n\to\infty} \left\| \left(\int_{\mathbb{T}} f_n(\theta,\cdot)^\alpha \ dm(\theta) \right)^{1/\alpha} \right\| = \left\| \left(\int_{\mathbb{T}} \|x(\cdot) + e^{i\theta} y(\cdot)\|^\alpha \ dm(\theta) \right)^{1/\alpha} \right\|.$$

Putting f_n instead of f in inequality (5.2) and taking a limit, we have

$$\left(\int_{\mathbb{T}} \|x + e^{i\theta} y\|_{E(X)}^\alpha \ dm \right)^{1/\alpha} \geq \left\| \left(\int_{\mathbb{T}} \|x(\cdot) + e^{i\theta} y(\cdot)\|^\alpha \ dm \right)^{1/\alpha} \right\|.$$

Hence letting $f, g \in E(X)$ with $\|f\| = 1$ and $\|g\| = 3^{1/\alpha}\epsilon > 0$, we get

$$\left(\int_{\mathbb{T}} \|f + e^{i\theta} g\|_{E(X)}^\alpha \ dm(\theta) \right)^{1/\alpha} \geq \left\| \left(\int_{\mathbb{T}} \|f(\cdot) + e^{i\theta} g(\cdot)\|_X^\alpha \ dm(\theta) \right)^{1/\alpha} \right\|_E.$$

Let

$$h(t) = \left(\int_{\mathbb{T}} \|f(t) + e^{i\theta} g(t)\|_X^\alpha \ dm(\theta) \right)^{1/\alpha}$$
$$A_1 = \{t : \|f(t)\| \geq \|g(t)\| \geq 0\}, \quad A_2 = \{t : \|f(t)\| = 0\},$$
$$A_3 = \{t : \|g(t)\| > \|f(t)\| > 0\}, \quad R = \text{support of } g.$$

Then $g = g\chi_{A_1} + g\chi_{A_2} + g\chi_{A_3}$. So there is $i = 1, 2, 3$ such that $\|g\chi_{A_i}\| \geq \epsilon$.

Case (1): Assume $\|g\chi_{A_1}\| \geq \epsilon$ and let

$$C = \{t : \|g(t)\| \geq \epsilon/3^{1/\alpha}\|f(t)\|\}.$$

Then

$$\begin{aligned} h(t) &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + h(t)\chi_{A_1\cap R}(t) \\ &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + h(t)\chi_{A_1\cap R\cap C}(t) + h(t)\chi_{A_1\cap R\setminus C}(t) \\ &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + \|f(t)\|_X(1 + H_1^X(\frac{\epsilon}{3^{1/\alpha}}))\chi_{A_1\cap R\cap C}(t) \\ &\quad + \|f(t)\|_X\chi_{A_1\cap R\setminus C}(t) \\ &\geq \|f(t)\|_X + H_1^X(\frac{\epsilon}{3^{1/\alpha}})\|f(t)\|_X\chi_{A_1\cap R\cap C}(t). \end{aligned}$$

Notice also that

$$\begin{aligned} \|f\chi_{A_1\cap R\cap C}\|_{E(X)}^\alpha &\geq \|g\chi_{A_1\cap R\cap C}\|_{E(X)}^\alpha = \|g\chi_{A_1\cap C}\|_{E(X)}^\alpha \\ &\geq \|g\chi_{A_1}\|_{E(X)}^\alpha - \|g\chi_{A_1\setminus C}\|_{E(X)}^\alpha \\ &\geq \|g\chi_{A_1}\|_{E(X)}^\alpha - \frac{\epsilon^\alpha}{3}\|f\chi_{A_1\setminus C}\|_{E(X)}^\alpha \geq \frac{2\epsilon^\alpha}{3}. \end{aligned}$$

Now the uniform monotonicity of E implies that

$$\begin{aligned}\|h\|_E &\geq \Big\|\|f(\cdot)\|_X + H_1^X\Big(\frac{\epsilon}{3^{1/\alpha}}\Big)\|f(\cdot)\|_X\chi_{A_1\cap R\cap C}\Big\|_E\\ &\geq 1+\Pi_1^E\Big(H_1^X\Big(\frac{\epsilon}{3^{1/\alpha}}\Big)\Big(\frac{2\epsilon^\alpha}{3}\Big)^{1/\alpha}\Big).\end{aligned}$$

Hence

$$\Big(\int_{\mathbb{T}}\|f+e^{i\theta}g\|_{E(X)}^\alpha\ dm(\theta)\Big)^{1/\alpha} \geq 1+\Pi_1^E\Big(H_1^X\Big(\frac{\epsilon}{3^{1/\alpha}}\Big)\Big(\frac{2\epsilon^\alpha}{3}\Big)^{1/\alpha}\Big).$$

Case (2): Assume $\|g\chi_{A_2}\|\geq\epsilon$. Then

$$\begin{aligned}h(t) &\geq \|f(t)\chi_{\Omega\setminus(A_2\cap R)}(t)\|_X + h(t)\chi_{A_2\cap R}(t)\\ &= \|f(t)\chi_{\Omega\setminus(A_2\cap R)}(t)\|_X + (\|f(t)\|_X+\|g(t)\|_X)\chi_{A_2\cap R}(t)\\ &= \|f(t)\|_X + \|g(t)\|_X\chi_{A_2}(t).\end{aligned}$$

It is clear that the uniform monotonicity of E implies that

$$\|h\|_E \geq 1+\Pi_1^E(\epsilon).$$

Hence

$$\Big(\int_{\mathbb{T}}\|f+e^{i\theta}g\|_{E(X)}^\alpha\ dm(\theta)\Big)^{1/\alpha} \geq 1+\Pi_1^E(\epsilon).$$

Case (3): Assume that $\|g\chi_{A_3}\|\geq\epsilon$. Then

$$h(t)\geq \|f(t)\|_X\chi_{\Omega\setminus A_3}(t)+h(t)\chi_{A_3}(t).$$

Let

$$\delta := 1-\Big(\frac{2+\Pi_1^E(\epsilon)}{2+2\Pi_1^E(\epsilon)}\Big)^\alpha > 0.$$

If $\|f\chi_{A_3}\|\leq\delta^{1/\alpha}$ then $\|f\chi_{\Omega\setminus A_3}\|^\alpha\geq 1-\delta$. Moreover

$$h(t)\geq \|f(t)\|_X\chi_{\Omega\setminus A_3}(t)+\|g(t)\|_X\chi_{A_3}(t).$$

Since the uniform monotonicity of E implies that

$$\|h\|_E\geq(1-\delta)^{1/\alpha}\big(1+\Pi_1^E(\epsilon)\big)=1+\frac{1}{2}\Pi_1^E(\epsilon),$$

so

$$\Big(\int_{\mathbb{T}}\|f+e^{i\theta}g\|_{E(X)}^\alpha\ d\theta\Big)^{1/\alpha} \geq 1+\frac{1}{2}\Pi_1^E(\epsilon).$$

If, on the other hand, $\|f\chi_{A_3}\| \geq \delta^{1/\alpha}$, then

$$\begin{aligned} h(t) &\geq \|f(t)\|_X \chi_{\Omega\setminus A_3}(t) + (1 + H_1^X(1))\|f(t)\|_X \chi_{A_3}(t) \\ &= \|f(t)\|_X + H_1^X(1)\|f(t)\|_X \chi_{A_3}(t). \end{aligned}$$

Thus by the uniform monotonicity of E,

$$\|h\|_E \geq 1 + \Pi_1^E(H_1^X(1)\delta^{1/\alpha}).$$

Hence

$$\left(\int_{\mathbb{T}} \|f + e^{i\theta} g\|_{E(X)}^\alpha \; dm(\theta)\right)^{1/\alpha} \geq 1 + \Pi_1^E(H_1^X(1)\delta^{1/\alpha}).$$

Combining these three cases and taking

$$\hat{\delta} = \min\left\{\Pi_1^E\left(H_1^X\left(\frac{\epsilon}{3^{1/\alpha}}\right)\left(\frac{2\epsilon^\alpha}{3}\right)^{1/\alpha}\right),\; \frac{1}{2}\Pi_1^E(\epsilon), \Pi_1^E(H_1^X(1)\delta^{1/\alpha})\right\},$$

we get

$$\left(\int_{\mathbb{T}} \|f + e^{i\theta} g\|_{E(X)}^\alpha \; d\theta\right)^{1/\alpha} \geq 1 + \hat{\delta},$$

which completes the proof. ∎

To conclude this paper we give some examples. Recall that, given $0 < p < \infty$ and a non-increasing, locally integrable function w: $[0, \gamma) \to (0, \infty)$, the **Lorentz space** $\Lambda_{p,w}$ is defined as follows

$$\Lambda_{p,w} = \left\{x \in L^0 : \|x\|_p = \left(\int_0^\gamma (x^*(t))^p w(t) \; dt\right)^{1/p} < \infty\right\},$$

where L^0 is a set of all measurable functions on $[0, \gamma)$ and x^* is a decreasing rearrangement of $x \in L^0$. For the definition and basic properties of decreasing rearrangement, see [2]. For $p = 1$ it is denoted by Λ_w. Observe that $\Lambda_{p,w}$ is a p-convexification of Λ_w. We say that the weight w is **regular** if $\inf_{t\in(0,\gamma)} S(t)/S(t/2) > 1$, where $S(t) = \int_0^t w(s) \; ds$.

Λ_w is uniformly monotone if and only if w is regular (from [16]). Hence Theorem 3.4 and Corollary 4.5 show the next proposition, which is a generalization of Corollary 3.6 in [5].

Proposition 5.4: *The following are equivalent:*

(1) *w is regular;*

(2) *Λ_w is uniformly monotone;*

(3) *$\Lambda_{p,w}$ is uniformly monotone for every $0 < p < \infty$; and*

(4) *the complex* $\Lambda_{p,w}$ *is uniformly* PL*-convex for every* $0 < p < \infty$

Using both Theorem 5.3 and Proposition 5.4 we get the next proposition, which is a generalization of Theorem 4.1 in [7].

PROPOSITION 5.5: *Suppose that* $(X, \|\cdot\|)$ *is a continuously quasi-normed space. Then the complex space* $\Lambda_{p,w}(X)$ *is uniformly* PL*-convex if and only if* X *is uniformly* PL*-convex and* w *is regular.*

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